Soliton excitations in a one-dimensional nonlinear diatomic chain of split-ring resonators

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We present a systematic analytical study of the dynamics of nonlinear magnetoinductive waves in a one-dimensional diatomic lattice of split ring resonators (SRRs) with Kerr nonlinear interaction between nearest neighbors. The linear spectrum of this model have two branches and exhibits a gap, which is proportional to the difference between two types of SRRs. We analyze the nonlinear excitations genuine of the discreteness and nonlinearity in such a diatomic chain based on an extended quasidiscreteness approach. Gap solitons (with vibrating frequency lying in the gap), resonant kinks (with the vibrating frequency lying in the frequency bands), and intrinsic localized modes (with the vibrating frequency being above all the frequency bands) are obtained explicitly. It is also shown that the existence of different localized structures depend strongly on the type of nonlinearity of the embedded medium (a self-focusing or defocusing one).

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I. INTRODUCTION

The pioneering works of Fermi, Pasta, and Ulam [1] and of Zabusky and Kruska [2] have stimulated a great variety of research on discrete solitons, or more generally localized nonlinear excitations in nonlinear discrete systems. For such discrete systems an accurate description involves a set of difference-differential equations and the intrinsic discreteness may drastically modify the nonlinear dynamics of the systems. The discreteness makes the properties of the systems periodic, so that due to the interplay between the discreteness and the nonlinearity, certain types of nonlinear excitations may exist, which have no counterpart whatsoever in continuous systems [3–6]. They are indeed natural modes of the systems. One of the typical and well-known examples is the crystal lattices subjected to a nonlinear on-site potential [7–9]. For the model of a diatomic chain there is a frequency gap between the acoustic and optical branches. In the linear case, a spectrum gap means that wave propagation of certain wavelength is forbidden, but when the nonlinearity is introduced, waves maybe allowed to propagate in the form of gap solitons. Other nonlinear excitations, such as resonant kinks, and intrinsic localized modes, have also been investigated.

Recently, artificially constructed metamaterials have become of considerable interest, because these materials can exhibit electromagnetic characteristic not available in naturally occurring materials [10]. Theoretical approaches to metamaterials often use an effective medium approximation, which relies on the averaging of microscopic fields [11,12]. The approximation is justified when the characteristic scale of the wavelength of the electromagnetic field is much larger than the period of the microstructured medium. Effective medium theory can be used to explain the phenomenon when electromagnetic wave enters metamaterials. However, such a theory, based on averaging, is bound to disregard the waves whose existence is entirely due to interaction between the individual elements through electromagnetic fields. Split ring resonators (SRRs) as one of typical resonant magnetic element have been attracted much attention since it was first proposed and applied to design negative refractive medium [11,13]. Such magnetic metamaterials (MMs) not only exhibit negative permeability but also support eigenmodes of the structure, which owe their existence to magnetic coupling between the elements [14,15]. The eigenmodes in the SRRs arrays are referred to as magnetoinductive (MI) waves. MI waves have been applied for delay lines [16], phase shifters [17], and microwave lenses [18] etc. Moreover, we note either by embedding the SRRs in a Kerr-type medium [19,20], or by inserting certain nonlinear elements (e.g., diodes) in each SRR [21–23], MMs may take on nonlinear properties. The combination of nonlinearity and discreteness in MMs arrays allows one to expect the formation of nonlinear localized structures. Recent researchs show one-dimensional (1D) or two-dimensional (2D) discrete array of a nonlinear monatomic chain of SRRs supports localized structures in the form of discrete breathers [24,25], magnetic domain walls [26], magnetoinductive envelope solitons [27], self-induced gap solitons [28]. Up to the present time, the study on nonlinear magnetoinductive wave focused on models of monatomic chains, the nonlinear excitations in diatomic chain has not been investigated to our knowledge.

In this paper, we consider the nonlinear excitations in 1D nonlinear diatomic chain of SRRs with nearest-neighbor interaction. Such 1D "magnetic diatomic" chain consists of two types of SRRs, and two branches of dispersion relation are expected which is analogous to the diatomic model of crystal lattice. If the nonlinearity of the system is considered, the characteristics of the spectrum changes and some localized nonlinear excitations may appear. To obtain the nonlinear excitations that vary slowly on the scale of the lattice spacing, we approximate discrete models by continuum partial differential equations, obtaining analytical solutions, which is close to the phenomena observed in original discrete (but often analytically intractable) models, using a systematic method called the quasidiscreteness approximation.

The paper is organized as follows. In Sec. II, we present our model, which describe the charge distribution in one-dimensional diatomic chain of SRRs with a Kerr interaction

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between nearest neighbors and the linear spectrum is also derived. In Sec. III, the asymptotic expansions are provided based on the quasidiscreteness technique. We derive two nonlinear envelope equations of amplitudes for acoustic and optical modes in this section. By using the results obtained in Sec. III, in Sec. IV, we present the analytical solutions of the nonlinear excitations at the band edge of the linear spectrum with zero-group velocity. Finally, Sec. V contains a discussion and summary of our results.

II. PHYSICAL MODEL

We consider a one-dimensional diatomic chain of SRRs with its nearest-neighbors interaction. The model is discrete and the separation of neighboring SRR centers is a being a fundamental physical parameter. A 1D array can be constructed either in the planar conjuration or in the axial conjuration [15]. In the simplest representation, each SRR is equivalent to an *RLC* oscillator. The rings form the inductances and the two slits as well as the gap between the two rings can be considered as capacitors. A magnetic field oriented perpendicular to the plane of the rings induces circulating currents owing to Faraday's law. The diatomic chain of SRRs includes two different size of SRRs, and we define self-inductance, Ohmic resistance, capacitance for the bigger SRR as L_1, R_1, C_1, and for the smaller SRR as L_2, R_2, C_2. The units become nonlinear due to Kerr dielectric that fills their gap [19] and equivalent permittivity is $\epsilon|E|^2=\epsilon_0(\epsilon_l+\alpha|E|^2/E_c^2)$ depending on the electric field E. The parameters ϵ_0 and ϵ_l represent the vacuum permittivity and linear permittivity respectively, E_c is a characteristic electric field, and $\alpha=+1$ ($\alpha=-1$) accounts for self-focusing (defocusing) nonlinearity. As a result, the SRRs acquire a field-dependent capacitance $C_{nj}|E|^2=\epsilon(|E_{gj}|^2)A_j/d_{gj}$, where $j=1,2$ accounts for two different kinds of SRRs. E_{gj} is the electric field induced along the SRR slit, A_j is the area of the cross section of the SRR wire, and d_{gj} is the size of the slit. If Q_n and q_n are the charge stored in the capacitor of the big and small SRR respectively, from the general relation of a voltage-dependent capacitance, we get,

$$Q_n = C_1\left(1+\alpha\frac{U_n^2}{3\epsilon_l U_{c1}^2}\right)U_n, \quad q_n = C_2\left(1+\alpha\frac{u_n^2}{3\epsilon_l U_{c2}^2}\right)u_n, \tag{1}$$

where $U_n=d_{g1}E_{g1}$ and $u_n=d_{g2}E_{g2}$ are the voltage across the slit of the SRRs, $C_j=\epsilon_0\epsilon_l(A_j/d_{gj})$ is the linear capacitance, and $U_{cj}=d_{gj}E_c$ ($j=1,2$). Neighboring SRRs are coupled due to magnetic dipole-dipole interaction through their mutual inductance M, which decays as the cube of the distance. For weak coupling between SRRs in a planar configuration, it is a good approximation to consider only nearest neighboring SRRs interactions. The dynamics of $Q_n(q_n)$ and the current $I_n(i_n)$ circulating in nth bigger SRR and smaller SRR can be described by

$$i_n = \frac{dq_n}{dt}, \tag{2}$$

$$I_n = \frac{dQ_n}{dt}, \tag{3}$$

$$L_1\frac{dI_n}{dt} + R_1 I_n + f(Q_n) = M\left(\frac{di_n}{dt}+\frac{di_{n+1}}{dt}\right) + \mathcal{E}_1(t), \tag{4}$$

$$L_2\frac{di_n}{dt} + R_2 i_n + f(q_n) = M\left(\frac{dI_n}{dt}+\frac{dI_{n-1}}{dt}\right) + \mathcal{E}_2(t), \tag{5}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are related to the electromotive force induced in each SRR due to the applied field. Making the translation: $t\to\omega_1 t$, $I_n\to I_c I_n$, $i_n\to i_c i_n$, $Q_n\to Q_c Q_n$, $q_n\to q_c q_n$, $\mathcal{E}_1\to\mathcal{E}_1 U_{c1}$, $\mathcal{E}_2\to\mathcal{E}_2 U_{c2}$, with $\omega_1^{-2}=L_1C_1$, $\omega_2^{-2}=L_2C_2$, $Q_c=C_1U_{c1}$, $q_c=C_2U_{c2}$, $I_C=U_{c1}\omega_1C_1$, $i_c=U_{c2}\omega_2C_2$, Eqs. (4) and (5) can be normalized to

$$\frac{d}{dt}(\lambda_1 i_n - I_n + \lambda_1 i_{n+1}) + f(Q_n) = \gamma_1 I_n + \mathcal{E}_1(t), \tag{6}$$

$$\frac{d}{dt}(\lambda_2 I_n - i_n + \lambda_2 I_{n+1}) + \lambda f(q_n) = \gamma_2 i_n + \mathcal{E}_2(t), \tag{7}$$

with $\lambda_1=Mi_c/(L_1I_c)$, $\lambda_2=MI_c/(L_2i_c)$, $\lambda=(I_cL_1U_{c1})/(i_cL_2U_{c2})$, $\gamma_1=R_1/(L_2\omega_1)$, $\gamma_2=R_2/(L_2\omega_1)$. γ_1,γ_2 are the loss coefficient, which is usually very small ($\gamma_1,\gamma_2\ll 1$), may account for both Ohmic and radiative losses. λ_1,λ_2 are the coupling parameters. λ denotes the difference between the big SRR and small SRR. Here, we define $\lambda>1$ without loss of generality. Making a Taylor expansion of Eq. (1) for $U_n=f(Q_n)$, $u_n=f(q_n)$ and keeping up to cubic terms, Eqs. (6) and (7) can be transformed to

$$\frac{d^2}{dt^2}(\lambda_1 q_n - Q_n + \lambda_1 q_{n+1}) - Q_n + \frac{\alpha}{3\epsilon_l}Q_n^3 = \gamma_1\frac{dQ_n}{dt} - \mathcal{E}_1(t), \tag{8}$$

$$\frac{d^2}{dt^2}(\lambda_2 Q_n - q_n + \lambda_2 Q_{n+1}) - \lambda q_n + \frac{\alpha}{3\epsilon_l}q_n^3 = \gamma_2\frac{dq_n}{dt} - \mathcal{E}_2(t). \tag{9}$$

The right-hand side will be omitted in the following, i.e., by setting $\gamma_1=\gamma_2=0$, $\mathcal{E}_1=\mathcal{E}_2=0$, thus neglecting losses and electromotive forcing. The linear dispersion relation of Eqs. (8) and (9) is

$$\omega_\pm^2 = \frac{(\lambda+1)\pm\{(\lambda+1)^2-4\lambda[1-2\lambda_1\lambda_2(1+\cos ka)]\}^{1/2}}{2-4\lambda_1\lambda_2(1+\cos ka)}, \tag{10}$$

with $-\pi/a<k\le\pi/a$. The minus (plus) sign corresponds to acoustic (optical) mode. We call $\omega_+(k)$ the acoustic and $\omega_-(k)$ the optical for comparison with the diatomic crystal lattices. The dispersion relations are depicted as two solid black curves in Fig. 1. At wave number $k=0$ the eigenfrequency spectrum has a lower cutoff $\omega_-^2(0)\equiv\omega_1^2=\{(\lambda+1)-[(\lambda-1)^2+16\lambda\lambda_1\lambda_2]^{1/2}\}/(2-8\lambda_1\lambda_2)$ for the acoustic mode and an upper cutoff $\omega_+^2(0)\equiv\omega_4^2=\{(\lambda+1)+[(\lambda-1)^2+16\lambda\lambda_1\lambda_2]^{1/2}\}/(2-8\lambda_1\lambda_2)$ for the optical mode. At $k=\pi/a$ there exists a fre-

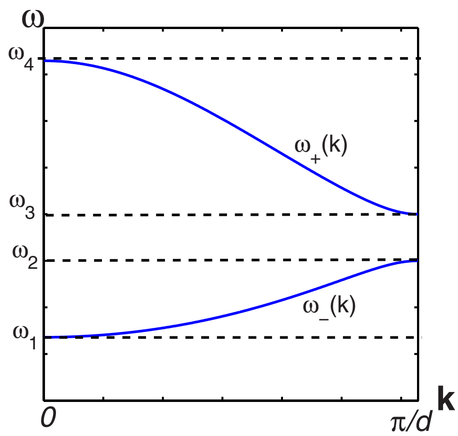

FIG. 1. (Color online) Dispersion curves in the diatomic chain of SRRs

quency gap between the upper cutoff of the acoustic branch, $\omega_-^2(\pi/a)\equiv\omega_2=1$, and the lower cutoff of the optical branch, $\omega_+^2(\pi/a)\equiv\omega_3=\lambda$. The width of the frequency gap is $\Delta\omega=\lambda-1$. We note the gap vanishes for the diatomic lattice when $\lambda\rightarrow 1$, thus, the system becomes continuously monatomic lattice and with increase of the difference between two type of SRRs, the separation between the acoustic and optical branches widens. In linear theory, the amplitudes of lattice waves are constants and linear waves cannot propagate and will be damped when the frequency ω lies in the regions $0<\omega<\omega_1$, $\omega_2<\omega<\omega_3$ and $\omega>\omega_4$. Accordingly, these regions are the "forbidden bands." However, when the nonlinearity in Eqs. (8) and (9) is considered, such waves maybe exist in the form of gap solitons or other localized modes. The oscillatory frequencies of localized modes can lie in the forbidden bands of the linear spectrum. The solutions of gap solitons and other localized modes may be obtained at $k=0$ or $k=\pm\pi/a$ as we will see in the following.

III. NONLINEAR ENVELOPE EQUATIONS FOR ACOUSTIC AND OPTICAL MODES

Because in general it is not possible to solve analytically nonlinear lattice equations such as Eqs. (8) and (9), some approximate theories have been developed. One powerful and clear-cut method is the method of quasidiscreteness approach, widely used in the study of nonlinear waves, solitons, and pattern formation in discrete system. The basic spirit of the quasidiscreteness approach is the assumption that a linear plane lattice wave is weakly modulated by the nonlinearity of the system. The carrier wave is discrete and can be described by a function with the fast variables n and t. The envelope is a function of the slow variables such as $\xi_n=\varepsilon(na-V_gt)$ and $\tau=\varepsilon^2t$. Here, V_g is a parameter to be determined by a solvability condition. ε $(0<\varepsilon\ll 1)$ is a small parameter denoting the relative amplitude of excitations. In this treatment one sets,

$$Q_n(t)=\varepsilon Q^{(1)}(\xi_n,\tau;\theta_n)+\varepsilon^2Q^{(2)}(\xi_n,\tau;\theta_n)+\varepsilon^3Q^{(3)}(\xi_n,\tau;\theta_n)+\ldots=\sum_{j=1}^{\infty}\varepsilon^jQ_{n,n}^{(j)}. \tag{11}$$

The fast variable, $\theta_n=kna-\omega t$, representing the phase of the carrier wave, is taken to be completely discrete. The same expansion for q_n is

$$q_n(t)=\sum_{j=1}^{\infty}\varepsilon^jq_{n,n}^{(j)}. \tag{12}$$

Substituting Eqs. (11) and (12) into Eqs. (8) and (9) and using the solvability conditions in the asymptotic expansion, we obtain the nonlinear Schrödinger equation (NLSE),

$$i\frac{\partial A_\pm}{\partial\tau}+\frac{1}{2}\Gamma_\pm\frac{\partial^2A_\pm}{\partial\xi_n^{\pm 2}}+\Delta_\pm|A_\pm|^2A_-=0, \tag{13}$$

where $A_+(\xi_n^+,\tau)$ and $A_-(\xi_n^-,\tau)$ are envelope functions for acoustic and optical modes respectively. The coefficients $\Gamma_\pm$, $\Delta_\pm$ in Eq. (13) are given in the Appendix. Letting $A_\pm(\xi_n^\pm,\tau)=(1/\varepsilon)F_\pm(x_n^\pm,\tau)$ and noting that $\xi_n^\pm=\varepsilon(na-V_g^\pm)=\varepsilon x_n^\pm(x_n^\pm=na-V_g^\pm)$ and $\tau=\varepsilon^2t$ with $V_g^\pm$ are group velocity for acoustic and optical modes shown in Appendix, Eq. (13) can be written as

$$i\frac{\partial F_\pm}{\partial t}+\frac{1}{2}\Gamma_\pm\frac{\partial^2F_\pm}{\partial x_n^{\pm 2}}+\Delta_\pm|F_\pm|^2F_\pm=0. \tag{14}$$

The NLSE [Eq. (14)] is completely integrable and can be solved by the inverse scattering method [29]. The results given above are valid in the whole Brillouin zone, $\pi/a<k\leq\pi/a$. Whether the solution of Eq. (14) is soliton or kink depends on the sign of $\Gamma_\pm\Delta_\pm$. The analytical solution will be given in the next section. The solutions of Eq. (14) describes modulations of the wave envelope in the reference frame moving at the group velocity $V_g^\pm$. We would like to note the two classes of localized structures when the wave number k approaches the edges of the Brillouin zone (0 or π/a) with vanishing group velocity. In the former case, i.e., $k=0$, the envelope $F_\pm(n,t)$ describes slow variations of the charge $q_n(Q_n)$ itself and in the latter case, the function $F_\pm(n,t)$ describes an envelope of the out of phase vibrations in the lattice.

IV. GAP SOLITONS, RESONANT KINKS, AND INTRINSIC LOCALIZED MODES

As we see for a diatomic chain of SRRs, there are two separate dispersion branches for the diatomic chain, upper branch $\omega_+(k)$ and lower branch $\omega_-(k)$ shown in Fig. 1. A gap exists between these two branches. Considering the nonlinearity in Eqs. (8) and (9), the oscillatory frequencies of localized modes can lie in the forbidden bands of the linear spectrum. In the following, we will write down the explicit expressions of nonpropagating solutions in the first-order approximation:

(1) Acoustic lower cutoff mode. For the acoustic mode at $k=0$, we have $\omega_-=\omega_1$ (the acoustic lower cutoff frequency). $V_g^-=0$, $x_n=na$, and

$$\Gamma_-=\frac{a^2\lambda_1\lambda_2\omega_1^5}{2\lambda-(1+\lambda)\omega_1^2}, \tag{15}$$

$$\Delta_- = \frac{\alpha\omega_1}{2\varepsilon_l}\frac{\lambda\lambda_2(-\omega_1^2+1)^2+\lambda_1(\omega_1^2-\lambda)^2}{\lambda_1(\lambda-\omega_1^2)[\lambda(1-\omega_1^2)+(\lambda-\omega_1^2)]}. \tag{16}$$

Considering the condition $\omega_1<1<\lambda$, Γ_- is always positive and the sign of Δ_- is determined by the parameter α. If the embedded medium is self-focusing, i.e., $\alpha>0$, the nonlinear parameter $\Delta_->0$. For sign$(\Gamma_-\Delta_-)>0$, Eq. (14) for F_- admits the soliton solution

$$F_-(x_n,t) = (\Gamma_-/\Delta_-)^{1/2}\eta_1 \operatorname{sech}[\eta_1(x_n-x_0)] \times \exp\left[i\left(\frac{1}{2}\Gamma_-\eta_1^2 t-\theta_0\right)\right], \tag{17}$$

where η_1, x_0, and θ_0 are arbitrary constants. Hence, in the first-order approximation we have

$$q_n = (\Gamma_-/\Delta_-)^{1/2}\eta_1 \operatorname{sech}[\eta_1(n-n_0)a] \times \exp[i(\Omega_1^s t-\theta_0)] + c.c., \tag{18}$$

$$Q_n = -2\omega_1^2\lambda_2/(\lambda-\omega_1^2)(\Gamma_-/\Delta_-)^{1/2}\eta_1 \operatorname{sech}[\eta_1(n-n_0)a] \times \exp[i(\Omega_1^s t-\theta_0)] + c.c. = -2\omega_1^2\lambda_2/(\lambda-\omega_1^2)q_n, \tag{19}$$

where n_0 is an arbitrary integer, and

$$\Omega_1^s = \omega_1 - \frac{1}{2}\Gamma_-\eta_1^2. \tag{20}$$

Ω_1 is less than ω_1, i.e., Ω_1 lies in the bottom gap of the dispersion curves. We call this type of nonlinear excitations an acoustic lower cutoff soliton or a bottom gap soliton. The central position of the localized mode is at site $n=n_0$, which depends on the initial exciting condition of the system.

When the embedded media is self-defocusing, $\alpha<0$, then sign$(\Gamma_-\Delta_-)<0$, a transition from soliton to kink occurs. Equation (14) for F_- admits the kink solution

$$F_-(x_n,t) = (\Gamma_-/|\Delta_-|)^{1/2}\eta_1 \tanh[\eta_1(x_n-x_0)] \times \exp\left[-i\left(\frac{1}{2}\Gamma_-\eta_1^2 t-\theta_0\right)\right], \tag{21}$$

Then the system has the configuration

$$q_n = (\Gamma_-/|\Delta_-|)^{1/2}\eta_1 \tanh[\eta_1(n-n_0)a] \times \exp[i(\Omega_1^k t-\theta_0)] + c.c., \tag{22}$$

$$Q_n = -2\omega_1^2\lambda_2/(\lambda-\omega_1^2)(\Gamma_-/|\Delta_-|)^{1/2}\eta_1 \tanh[\eta_1(n-n_0)a] \times \exp[i(\Omega_1^k t-\theta_0)] + c.c. = -2\omega_1^2\lambda_2/(\lambda-\omega_1^2)q_n, \tag{23}$$

with

$$\Omega_1^k = \omega_1 + \frac{1}{2}\Gamma_-\eta_1^2. \tag{24}$$

being within the frequency band of the acoustic mode, so it is a resonant kink of the system. Equations (18), (19), (22), and (23) show that the vibration of two sublattices are out of phase and with different amplitudes.

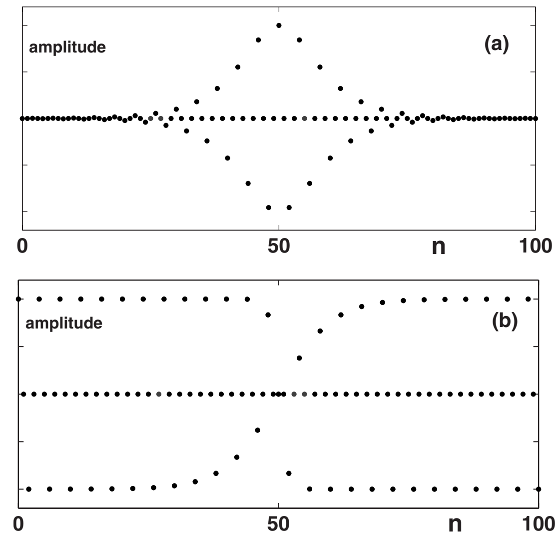

FIG. 2. The pattern of the gap solitons in 1D diatomic lattice of SRRs: (a) acoustic upper cutoff mode; (b) optical lower cutoff mode.

(2) Acoustic upper cutoff mode. For the acoustic mode at $q=\pm\pi/a$, we have $\omega_-=\omega_2$ (the acoustic upper cutoff frequency), $V_g^-=0$, and

$$\Gamma_- = -\frac{a^2\lambda_1\lambda_2}{\lambda-1}, \quad \Delta_- = \frac{\alpha}{2\varepsilon_l}. \tag{25}$$

$\Gamma_-<0$ and the sign of Δ_- is determined by the type of media α. For sgn $(\Gamma_-\Delta_->0)$, one has the single soliton solution

$$F_-(x_n,t) = (|\Gamma_-|/\Delta_-)^{1/2}\eta_2 \operatorname{sech}[\eta_2(x_n-x_0)] \times \exp\left[-i\left(\frac{1}{2}|\Gamma_-|\eta_2^2 t-\theta_0\right)\right], \tag{26}$$

where η_2 is arbitrary constants. Then the system has the configuration

$$q_n = (-1)^n(|\Gamma_-|/\Delta_-)^{1/2}\eta_2 \operatorname{sech}[\eta_2(n-n_0)a] \times \exp[i(\Omega_2^s t-\theta_0)] + c.c., \tag{27}$$

$$Q_n = 0, \tag{28}$$

with

$$\Omega_2^s = \omega_2 + \frac{1}{2}|\Gamma_-|\eta_2^2, \tag{29}$$

being within the frequency gap between the dispersion curves of the acoustic and optical modes. It is a typical gap soliton when the condition $\alpha<0$ is satisfied, i.e., the embedded medium is self-defocusing. The vibrating frequency Ω_2^s has the parabola relation with respect to the wave amplitude, denoted by the parameter η_2. It is the nonlinearity of the system that induces localization in frequency gap. In these localized modes, all the big SRRs have no current and the small ones form a nonpropagating envelope soliton with opposite phase between the nearest neighbors. The lattice vibratory patterns are shown in Fig. 2(a).

For sgn $(\Gamma_-\Delta_-<0)$, one has the kink soliton solution,

$$F_-(x_n,t) = (|\Gamma_-|/\Delta_-)^{1/2}\eta_2 \tanh[\eta_2(x_n - x_0)] \times \exp\left[i\left(\frac{1}{2}|\Gamma_-|\eta_2^2 t - \theta_0\right)\right], \quad (30)$$

where η_2 is arbitrary constants. The charge distribution is

$$q_n = (-1)^n(|\Gamma_-|/\Delta_-)^{1/2}\eta_2 \tanh[\eta_2(n-n_0)a] \times \exp[i(\Omega_2^k t + \theta_0)] + c.c., \quad (31)$$

$$Q_n = 0, \quad (32)$$

with

$$\Omega_2^k = \omega_2 - \frac{1}{2}|\Gamma_-|\eta_2^2, \quad (33)$$

which is within the frequency band of the acoustic mode. It can be called as acoustic upper cutoff resonant kink. There is always no charge distribution in the big SRRs and in the small ones the charge distribution vary with opposite phase.

(3) Optical lower cutoff mode. For the optical mode at $k=0$, one has $\omega_+=\omega_3$ (the optical lower cutoff frequency), $V_g=0$, and

$$\Gamma_+ = \frac{a^2\lambda_1\lambda_2\omega_3^5}{2\lambda - (1+\lambda)\omega_3}, \quad (34)$$

$$\Delta_+ = \frac{\alpha\omega_3}{2\varepsilon_l}\frac{\lambda\lambda_2(-\omega_3^2+1)^2 + \lambda_1(\omega_3^2-\lambda)^2}{\lambda_2(1-\omega_3^2)[\lambda(1-\omega_3^2)+(\lambda-\omega_3^2)]}. \quad (35)$$

Γ_+ is always negative and Δ_+ is also determined by α. For sgn $(\Gamma_-\Delta_- > 0)$, one has the single soliton solution

$$F_+(x_n,t) = (|\Gamma_+|/|\Delta_+|)^{1/2}\eta_3 \operatorname{sech}[\eta_3(x_n - x_0)] \times \exp\left[-i\left(\frac{1}{2}|\Gamma_+|\eta_3^2 t - \theta_0\right)\right], \quad (36)$$

where η_3 is arbitrary constants. The charge distribution in this case takes the form,

$$Q_n = (-1)^n(|\Gamma_+|/|\Delta_+|)^{1/2}\eta_3 \operatorname{sech}[\eta_3(n-n_0)a] \times \exp[i(\Omega_3^s t - \theta_0)] + c.c., \quad (37)$$

$$q_n = 0, \quad (38)$$

with

$$\Omega_3^s = \omega_3 + \frac{1}{2}|\Gamma_+|\eta_3^2, \quad (39)$$

which is within the frequency band of the optical branch. At this mode, the small SRRs has no current, and the big SRRs oscillate with opposite phase between their nearest neighbors, forming a nonpropagating envelope soliton.

For sgn $(\Gamma_+\Delta_+ < 0)$, one has the kink soliton solution

$$F_+(x_n,t) = (|\Gamma_+|/\Delta_+)^{1/2}\eta_3 \tanh[\eta_3(x_n - x_0)] \times \exp\left[i\left(\frac{1}{2}|\Gamma_+|\eta_3^2 t - \theta_0\right)\right], \quad (40)$$

where η_3 is arbitrary constants. we have

$$Q_n = (-1)^n(|\Gamma_+|/\Delta_+)^{1/2}\eta_3 \tanh[\eta_3(n-n_0)a] \times \exp[i(\Omega_3^k t - \theta_0)] + c.c., \quad (41)$$

$$q_n = 0, \quad (42)$$

with

$$\Omega_3^k = \omega_3 - \frac{1}{2}|\Gamma_-|\eta_3^2, \quad (43)$$

being within the frequency gap between the dispersion curves of the acoustic and optical modes. We can see such a diatomic chain of SRRs may support the localized mode in the form of kinks provided the nonlinear vibrate frequency lies in the gap. The lattice vibratory patterns are shown in Fig. 2(b).

(4) Optical upper cutoff mode. For the optical mode at $k=\pm 0$, we have $\omega_+=\omega_4$ (the optical upper cutoff frequency), $V_g=0$, and

$$\Gamma_+ = -\frac{a^2\lambda_1\lambda_2\omega_4^5}{2\lambda - (1+\lambda)\omega_4}, \quad \Delta_+ = \frac{\alpha}{2\varepsilon_l}\sqrt{\lambda}. \quad (44)$$

Due to $\omega_4^2 > \lambda > 2\lambda/(1+\lambda)$, $2\lambda-(1+\lambda)\omega_4 < 0$, $\Gamma_+ > 0$, and the sign of Δ_+ is also decided by the type of the embedding medium. For sgn $(\Gamma_+\Delta_+ > 0)$, one has the single soliton solution

$$F_+(x_n,t) = (\Gamma_+/\Delta_+)^{1/2}\eta_4 \operatorname{sech}[\eta_4(x_n - x_0)] \times \exp\left[i\left(\frac{1}{2}\Gamma_+\eta_4^2 t - \theta_0\right)\right], \quad (45)$$

where η_4 is arbitrary constants. The lattice configuration takes the form,

$$Q_n = (\Gamma_+/\Delta_+)^{1/2}\eta_4 \operatorname{sech}[\eta_4(n-n_0)a] \times \exp[i(\Omega_4^s t - \theta_0)] + c.c., \quad (46)$$

$$q_n = 2\lambda_1\omega_4^2/(\omega_4^2-1)(\Gamma_+/\Delta_+)^{1/2}\eta_2 \operatorname{sech}[\eta_4(n-n_0)a] \times \exp[i(\Omega_4 t - \theta_0)] + c.c. = 2\lambda_1\omega_4^2/(\omega_4^2-1)Q_n, \quad (47)$$

with

$$\Omega_4^s = \omega_4 - \frac{1}{2}\Gamma_+\eta_4^2, \quad (48)$$

being within the frequency band of the optical mode. The two kinds of SRRs oscillate with the same phase and different amplitude.

For sgn $(\Gamma_-\Delta_- < 0)$, one has the kink soliton solution

$$F_+(x_n,t) = (\Gamma_+/|\Delta_+|)^{1/2}\eta_4 \tanh[\eta_4(x_n - x_0)] \times \exp\left[-i\left(\frac{1}{2}\Gamma_+\eta_4^2 t - \theta_0\right)\right], \quad (49)$$

where η_4 is arbitrary constants. Hence, in the first-order approximation we have

$$Q_n = (\Gamma_+/|\Delta_+|)^{1/2}\eta_4 \tanh[\eta_4(n-n_0)a] \times \exp[-i(\Omega_4^k t - \theta_0)] + c.c., \quad (50)$$

$$q_n = 2\lambda_1\omega_4^2/(\omega_4^2-1)\Gamma_+/|\Delta_+|)^{1/2}\eta_4 \tanh[\eta_4(n-n_0)a] \times \exp[-i(\Omega_4^k t - \theta_0)] + c.c. = 2\lambda_1\omega_4^2/(\omega_4^2-1)Q_n, \quad (51)$$

with

$$\Omega_4^k = \omega_4 + \frac{1}{2}\Gamma_+\eta_4^2, \quad (52)$$

i.e., the vibrating frequency of the localized mode is above the spectrum of the linear optical mode. So Eqs. (50) and (51) is an intrinsic localized mode of the system with kink-type solution. Two types of SRR have the same phase.

Noticing that in the experimental 1D diatomic SRRs array system, the self-inductance of two different kinds of circular SRR is determined by the $L_j = \mu_0 w_j[\ln(16w_j/h_j) - 1.75](j=1,2)$ which the parameter w_j is radius and h_j is diameter of the circular cross section. The expression for the mutual inductance between two SRRs can be calculated by means of a simple approximation as $M \simeq \mu_0 \pi w_1^2 w_2^2/4a^3$ for planar geometry. For axial geometry the mutual inductance is $M \simeq \mu_0\pi w_1^2 w_2^2/2a^3$. The coupling parameter $\lambda_j(j=1,2)$ calculated in the axial geometry is twice as strong as planar geometry with the same separation distance a approximatively. Choosing $w1=2.5$ μm, $h1=1.8$ μm, $d_{g1}=1$ μm and $w2=2.4$ μm, $h2=2$ μm, $d_{g2}=1$ μm with the separation distance as $a=5.1$ μm, the resonance frequency for one SRR is about 16.3 THz, and for another is about 15.8 THz [24,25]. For axial geometry the coupling parameters are $\lambda_1=0.12$, $\lambda_2=0.15$ and the important parameter λ relating to the gap size can be calculated as 1.2. It is appropriate to choose the linear permittivity $\epsilon_l=2$. The above requirements can be easily realized in experiment. The great flexibility of metamaterial engineering maybe provide an opportunity to observe different kinds of nonlinear excitations in such system.

V. SUMMARY AND DISCUSSION

In summary, we have analytically studied the nonlinear dynamics in a 1D diatomic lattice of SRRs with Kerr nonlinearity by use of the method of quasidiscreteness approximation. Many different types of nonlinear excitations have been obtained in a unified way.

We first present the discrete model of 1D diatomic chain of SRRs subjected to Kerr nonlinearity considering only nearest neighboring interactions. For such discrete systems an accurate description involves a set of difference-differential equations and the intrinsic discreteness combined with nonlinearity can drastically modify the dynamics of the system. The nonlinear controlling equations describing the charge variation is derived and normalized. There is an important parameter λ to denote the difference between two types of SRRs. The linear dispersion relation is obtained and a frequency gap appear between the acoustic and optical branches. The gap size is decided by the parameter λ, and when $\lambda \to 1$, i.e., the gap disappear.

In Sec. III, using an asymptotic expansion for the charge distribution of each element under a quasidiscreteness approximation, the original nonlinear equations can be transformed into a set of inhomogeneous linear equations, which can be solved order by order. The expansion procedure is quite general and can be applied to other nonlinear lattice systems. We have solved the acoustic and the optical modes, respectively. For excitations that vary slowly on the scale of the lattice spacing, one can approximate discrete models by continuum partial differential NLS equations in the whole Brillouin zone. In Sec. IV, the nonlinear localized modes relevant to the cutoff modes of the linear bands were obtained. Our results show that, for 1D diatomic chain of SRRs, one has different types of nonlinear localized modes, such as the bottom gap soliton, the resonant kink, the gap soliton, and intrinsic localized mode. We note the existence of different localized structures depend drastically on the type of nonlinearity (a self-focusing or defocusing one). Gap solitons are our interesting nonlinear excitations in diatomic lattice systems. We analyze soliton solutions with the frequencies lying in the vicinity of the gap of the linear spectrum. It is shown analytically that such a diatomic chain of SRRs may support the gap solitons provided the soliton frequency lies within the gap. There are two type of gap solitons. One is an envelope soliton shown in Eq. (27), and Eq. (41) is the other one which is a kink-type soliton.

Considering magnetic metamaterials composed of SRR can be controlled by adjusting the size and distance between them artificially, a discrete array of nonlinear SRRs is a good system to display a richer class nonlinear excitations. Up to now, the dynamics of nonlinear magnetoinductive wave has never been studied in 1D diatomic chain of SRRs. We believe our results may be useful to help further understand the excitation spectrum and properties in magnetic metamaterials as well as be a guide for experimental findings.

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APPENDIX: DERIVATION OF THE NLSE

Using the expansion in Eqs. (11) and (12) and equating the coefficients of the same powers of ε from Eqs. (8) and (9), we obtain

$$\frac{\partial^2}{\partial t^2}(\lambda_1 q_{n,n}^{(j)} - Q_{n,n}^{(j)} + \lambda_1 q_{n,n+1}^{(j)}) - Q_{n,n}^{(j)} = M_n^{(j)}, \quad (A1)$$

with

$$M_n^{(1)} = 0, \quad (A2)$$

$$M_n^{(2)} = \omega^2\lambda_1 a \frac{\partial}{\partial \xi_n} Q_{n,n+1}^{(1)} - 2i\omega V_g \frac{\partial}{\partial \xi_n}(\lambda_1 Q_{n,n}^{(1)} - q_{n,n}^{(1)} + \lambda_1 Q_{n,n+1}^{(1)}), \quad (A3)$$

$$M_n^{(3)} = \omega^2\left[\lambda_1\left(\frac{1}{2}a^2\frac{\partial^2}{\partial\xi_n^2}Q_{n,n+1}^{(1)} - a\frac{\partial}{\partial\xi_n}Q_{n,n+1}^{(2)}\right)\right] + \frac{\alpha}{3\varepsilon_l}q_{n,n}^{(1)3}, \tag{A4}$$

$$- 2i\omega V_g\frac{\partial}{\partial\xi_n}\left[\lambda_1 Q_{n,n}^{(2)} - q_{n,n}^{(2)} + \lambda_1\left(a\frac{\partial}{\partial\xi_n}Q_{n,n+1}^{(1)} + Q_{n,n+1}^{(2)}\right)\right] + 2i\omega\frac{\partial}{\partial\tau}(\lambda_1 Q_{n,n}^{(1)} - q_{n,n}^{(1)} + \lambda_1 Q_{n,n+1}^{(1)}) - 2i\omega V_g^2\frac{\partial^2}{\partial\xi_n^2}(\lambda_1 Q_{n,n}^{(1)} - q_{n,n}^{(1)} + Q_{n,n+1}^{(1)}), \tag{A5}$$

and

$$\frac{\partial^2}{\partial t^2}(\lambda_2 Q_{n,n}^{(j)} - q_{n,n}^{(j)} + \lambda_2 Q_{n,n-1}^{(j)}) - \lambda q_{n,n}^{(j)} = N_n^{(j)}, \tag{A6}$$

with

$$N_n^{(1)} = 0, \tag{A7}$$

$$N_n^{(2)} = -\omega^2\lambda_2 a\frac{\partial}{\partial\xi_n}q_{n,n-1}^{(1)} + 2i\omega V_g\frac{\partial}{\partial\xi_n}(\lambda_2 q_{n,n}^{(1)} - Q_{n,n}^{(1)} + \lambda_2 q_{n,n-1}^{(1)}), \tag{A8}$$

$$N_n^{(3)} = \omega^2\left[\lambda_2\left(\frac{1}{2}a^2\frac{\partial^2}{\partial\xi_n^2}q_{n,n-1}^{(1)} + a\frac{\partial}{\partial\xi_n}q_{n,n-1}^{(2)}\right)\right] + \lambda\frac{\alpha}{3\varepsilon_l}Q_{n,n}^{(1)3} \tag{A9}$$

$$- 2i\omega V_g\frac{\partial}{\partial\xi_n}\left[\lambda_2 q_{n,n}^{(2)} - Q_{n,n}^{(2)} + \lambda_2\left(-a\frac{\partial}{\partial\xi_n}Q_{n,n-1}^{(1)} + Q_{n,n-1}^{(2)}\right)\right] + 2i\omega\frac{\partial}{\partial\tau}(\lambda_2 q_{n,n}^{(1)} - Q_{n,n}^{(1)} + \lambda_2 q_{n,n-1}^{(1)}) - 2i\omega V_g^2\frac{\partial^2}{\partial\xi_n^2}(\lambda_2 q_{n,n}^{(1)} - Q_{n,n}^{(1)} + q_{n,n-1}^{(1)}). \tag{A10}$$

We rewrite Eqs. (A1) and (A6) in the form,

$$\lambda_1\lambda_2\hat{L}_0^2(2Q_{n,n}^{(j)} + Q_{n,n+1}^{(j)} + Q_{n,n-1}^{(j)}) - \hat{L}_1\hat{L}_2 Q_{n,n}^{(j)} = \hat{L}_1 N_n^{(j)} + \lambda_2\hat{L}_0(M_n^{(j)} + M_{n-1}^{(j)}), \tag{A11}$$

$$\lambda_1\lambda_2\hat{L}_0^2(2q_{n,n}^{(j)} + q_{n,n+1}^{(j)} + q_{n,n-1}^{(j)}) - \hat{L}_1\hat{L}_2 q_{n,n}^{(j)} = \hat{L}_2 M_n^{(j)} + \lambda_1\hat{L}_0(N_n^{(j)} + N_{n+1}^{(j)}), \tag{A12}$$

with $\hat{L}_0=\partial^2/\partial t^2$, $\hat{L}_1=1+\partial^2/\partial t^2$, $\hat{L}_2=\lambda+\partial^2/\partial t^2$ and $j=1,2,3,\ldots$ One can solve the Eq. (A11) to obtain the charge variation $Q_{n,n}^{(j)}$ of big SRRs, and get the charge variation of small ones, $q_{n,n}^{(j)}$, from Eq. (A12) step by step using the solvability conditions in the asymptotic expansion.

1. ACOUSTIC MODE EXCITATIONS

First, we investigate the lower-frequency (acoustic) mode of the system. For $j=1$ we have the linear equations from Eqs. (A11) and (A12),

$$\lambda_1\lambda_2\hat{L}_0^2(2Q_{n,n}^{(1)} + Q_{n,n+1}^{(1)} + Q_{n,n-1}^{(1)}) - \hat{L}_1\hat{L}_2 Q_{n,n}^{(1)} = 0, \tag{A13}$$

$$\lambda_1\lambda_2\hat{L}_0^2(2q_{n,n}^{(1)} + q_{n,n+1}^{(1)} + q_{n,n-1}^{(1)}) - \hat{L}_1\hat{L}_2 q_{n,n}^{(1)} = 0. \tag{A14}$$

It is easy to get the solution

$$q_{n,n}^{(1)} = A_-(\xi_n^-,\tau)\exp(i\theta_n^-) + c.c., \tag{A15}$$

$$Q_{n,n}^{(1)} = \frac{\lambda_2\omega_-^2[1+\exp(-ika)]}{\omega_-^2 - \lambda}A_-(\xi_n^-,\tau)\exp(i\theta_n^-) + c.c., \tag{A16}$$

with $\theta_n^-=kna-\omega_- t$ and c.c.. represent the complex conjugate. $\omega_-(k)$ has been given in Eq. (10) with a minus sign. The envelope function $A_-(\xi_n,\tau)$ is yet to be determined. For $j=2$, Eqs. (A11) and (A12) take the form,

$$\lambda_1\lambda_2\hat{L}_0^2(2Q_{n,n}^{(2)} + Q_{n,n+1}^{(2)} + Q_{n,n-1}^{(2)}) - \hat{L}_1\hat{L}_2 Q_{n,n}^{(2)} = \hat{L}_1 N_n^{(2)} + \lambda_2\hat{L}_0(M_n^{(2)} + M_{n-1}^{(2)}), \tag{A17}$$

$$\lambda_1\lambda_2\hat{L}_0^2(2q_{n,n}^{(2)} + q_{n,n+1}^{(2)} + q_{n,n-1}^{(2)}) - \hat{L}_1\hat{L}_2 q_{n,n}^{(2)} = \hat{L}_2 M_n^{(2)} + \lambda_1\hat{L}_0(N_n^{(2)} + N_{n+1}^{(2)}). \tag{A18}$$

Using Eqs. (A15) and (A16), we can calculate $M_n^{(2)}$, $M_{n-1}^{(2)}$, $N_n^{(2)}$ and $N_{n+1}^{(2)}$. Then we have,

$$\lambda_1\lambda_2\hat{L}_0^2(2q_{n,n}^{(2)} + q_{n,n+1}^{(2)} + q_{n,n-1}^{(2)}) - \hat{L}_1\hat{L}_2 q_{n,n}^{(2)} = \{[2\lambda - (\lambda+1)\omega_-^2]V_g^- + 2\lambda_1\lambda_2 a\omega_-^5\sin(ka)\}\frac{\partial A_-}{\partial\xi_n^-}e^{i\theta_n} + c.c. \tag{A19}$$

Since the term proportional to $\exp(i\theta_n^-)$ on the right-hand side of Eq. (A19) is a secular term it must be eliminated (solvability condition). One has

$$V_g^- = \frac{\lambda_1\lambda_2 a\omega_-^5\sin(ka)}{2\lambda - (\lambda+1)\omega_-^2} = \frac{d\omega_-}{dk}, \tag{A20}$$

i.e., it is the group velocity of the carrier wave. Solving Eqs. (A17) and (A18) under the condition (A20) we obtain

$$Q_{n,n}^{(2)} = B_-(\xi_n^-,\tau)\exp(i\theta_n^-) + c.c., \tag{A21}$$

$$q_{n,n}^{(2)} = \frac{1}{\omega_-^2 - 1}\left\{\lambda_1\omega_-^2(1+e^{ika})B_- + \left[\lambda_1\omega_-^2 a e^{ika} - \frac{2i\lambda_1\omega_-(1+e^{ika})}{\omega_-^2}V_g^-\right]\frac{\partial A_-}{\partial\xi_n^-}\right\}\exp(i\theta_n^-) + c.c., \tag{A22}$$

where B_- is another undetermined function. In fact, we can set $B_-=0$ because it may be transferred to the lowest-order solution Eqs. (A13) and (A14) and the transferred quantity can be regarded as a new expression for A_- [30]. Thus, we have,

$$Q_{n,n}^{(2)} = 0, \tag{A23}$$

$$q_{n,n}^{(2)} = \frac{1}{\omega_-^2 - 1}\left[\lambda_1 \omega_-^2 a e^{ika} - \frac{2i\lambda_1\omega_-(1+e^{ika})}{\omega_-^2} V_g^-\right]\frac{\partial A_-}{\partial \xi_n^-}\exp(i\theta_n^-) + c.c., \tag{A24}$$

In the order $j=3$, we have the third-order approximate equations,

$$\lambda_1\lambda_2\hat{L}_0^2(2Q_{n,n}^{(3)} + Q_{n,n+1}^{(3)} + Q_{n,n-1}^{(3)}) - \hat{L}_1\hat{L}_2 Q_{n,n}^{(3)} = \hat{L}_1 N_n^{(3)} + \lambda_2\hat{L}_0(M_n^{(3)} + M_{n-1}^{(3)}), \tag{A25}$$

$$\lambda_1\lambda_2\hat{L}_0^2(2q_{n,n}^{(3)} + q_{n,n+1}^{(3)} + q_{n,n-1}^{(3)}) - \hat{L}_1\hat{L}_2 q_{n,n}^{(3)} = \hat{L}_2 M_n^{(3)} + \lambda_1\hat{L}_0(N_n^{(3)} + N_{n+1}^{(3)}). \tag{A26}$$

By use of Eqs. (A15), (A16), (A21), and (A22), we can get $M_n^{(3)}$, $M_{n-1}^{(3)}$, $N_n^{(3)}$, and $N_{n+1}^{(3)}$. A detailed calculation yields,

$$\lambda_1\lambda_2\hat{L}_0^2(2Q_{n,n}^{(3)} + Q_{n,n+1}^{(2)} + Q_{n,n-1}^{(3)}) - \hat{L}_1\hat{L}_2 Q_{n,n}^{(3)} = \left[\lambda_2\omega_-^2(1+e^{-ika})\left(i\frac{\partial A_-}{\partial \tau} + \frac{1}{2}\Gamma_-\frac{\partial^2 A_-}{\partial \xi_n^{-2}} + \Delta_-|A_-|^2 A_-\right)\right]e^{i\theta_n^-} + c.c., \tag{A27}$$

with

$$\Gamma_- = \frac{2(1+\lambda)\omega_- V_g^{-2} + 5a\lambda_1\lambda_2\sin(ka)\omega_-^4 V_g^- + a^2\lambda_1\lambda_2\omega_-^5\cos(ka)}{2\lambda - (1+\lambda)\omega_-^2}, \tag{A28}$$

$$\Delta_- = \frac{\alpha\omega_-}{2\varepsilon_l}\frac{\lambda\lambda_2(-\omega_-^2+1)^2 + \lambda_1(\omega_-^2-\lambda)^2}{\lambda_1(\lambda-\omega_-^2)[\lambda(1-\omega_-^2)+(\lambda-\omega_-^2)]}. \tag{A29}$$

Again the solvability condition for $Q_{n,n}^{(3)}$ requires that the coefficient proportional to $\exp(i\theta_n^-)$ on the right-hand side of Eq. (A27) vanishes. This gives the closed equation for $A_-(\xi_n,\tau)$,

$$i\frac{\partial A_-}{\partial \tau} + \frac{1}{2}\Gamma_-\frac{\partial^2 A_-}{\partial \xi_n^{-2}} + \Delta_-|A_-|^2 A_- = 0. \tag{A30}$$

2. OPTICAL MODE EXCITATIONS

Second, we consider the higher-frequency optical-mode excitations. We can solve them order by order by the procedure used in solving the acoustic mode in the last subsection. Thus, we can write the solution of the optical mode. For $j=1$, we have

$$Q_{n,n}^{(1)} = A_+(\xi_n^+,\tau)\exp(i\theta_n^+) + c.c., \tag{A31}$$

$$q_{n,n}^{(1)} = \frac{\lambda_1\omega_+^2[1+\exp(ika)]}{\omega_+^2 - 1}A_+(\xi_n^+,\tau)\exp(i\theta_n^+) + c.c., \tag{A32}$$

where $\theta_n^+(t) = kna - \omega_+(k)t$ and $A_+(\xi_n^+,\tau)$ is an envelope function yet to be determined. $\omega_+(k)$ is the dispersion relation for the optical mode, given in Eq. (10) with a plus sign. In the order $j=2$, the solvability condition for $Q_{n,n}^{(2)}$ gives

$$V_g^+ = \frac{\lambda_1\lambda_2 a\omega_+^5\sin(ka)}{2\lambda - (\lambda+1)\omega_+^2} = \frac{d\omega_+}{dk}, \tag{A33}$$

so $\xi_n = \xi_n^+ = \epsilon(na - V_g^+ t)$. The second-order solution reads

$$Q_{n,n}^{(1)} = 0, \tag{A34}$$

$$q_{n,n}^{(1)} = \frac{\lambda_1\omega_+^2[1+\exp(ika)]}{\omega_+^2 - 1}A_+(\xi_n^+,\tau)\exp(i\theta_n^+) + c.c. \tag{A35}$$

For $j=3$ the solvability condition for $Q_{n,n}^{(3)}$ yields the evolution equation for A_+,

$$i\frac{\partial A_+}{\partial \tau} + \frac{1}{2}\Gamma_+\frac{\partial^2 A_+}{\partial \xi_n^{+2}} + \Delta_+|A_+|^2 A_+ = 0, \tag{A36}$$

with

$$\Gamma_+ = \frac{2(1+\lambda)\omega_+ V_g^{+2} + 5a\lambda_1\lambda_2\sin(ka)\omega_+^4 V_g^+ + a^2\lambda_1\lambda_2\omega_+^5\cos(ka)}{2\lambda - (1+\lambda)\omega_+^2}, \tag{A37}$$

$$\Delta_+ = \frac{\alpha\omega_+}{2\varepsilon_l}\frac{\lambda\lambda_2(-\omega_+^2+1)^2 + \lambda_1(\omega_+^2-\lambda)^2}{\lambda_2(1-\omega_+^2)[\lambda(1-\omega_+^2)+(\lambda-\omega_+^2)]}, \tag{A38}$$

where $\omega_\pm(k)$ has been given in Eq. (10).

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